Consider the inertial coordinate system \( xyz \) with origin \( O \) and the non-inertial coordinate system \( x'y'z' \) with origin \( O' \). The \( x'y'z' \) coordinate system is allowed to translate and rotate at an angular velocity \( \Omega \). A particle \( P \) can be identified with respect to the non-inertial coordinate system by the position vector \( \vec{r} \) and with respect to the inertial coordinate system by the position vector \( \vec{\eta} \). The position vector \( \vec{r} \) can be written in terms of the non-inertial coordinate system as,
\[
\vec{r} = x'i + y'j + z'k ,
\]
and the position vector of \( P \) in the inertial frame is related to the non-inertial position vector by
\[
\vec{\eta} = \vec{\xi} + \vec{r} .
\] (1)

The velocity of with respect to the inertial coordinate system is obtained by differentiating (1) with respect to time.
\[
\vec{V}_{xyz} = \frac{d\vec{\eta}}{dt} = \frac{d\vec{\xi}}{dt} + \frac{d\vec{r}}{dt} = \vec{V}_{\Omega} + \frac{d\vec{r}}{dt}
\] (2)

where, \( \vec{V}_{\Omega} \) is the velocity of the non-inertial frame origin. The unit vectors \( \hat{i} \) and \( \hat{j} \) that describe \( \vec{r} \) are not constant (although there magnitude is) hence, differentiation of \( \vec{r} \) must include differentiating the unit vectors with respect to time also. Doing this yields,
Recall from the discussion in class of the centrifugal force that for a constant magnitude vector $\vec{R}$ rotating: $\frac{d\vec{R}}{dt} = \vec{\omega} \times \vec{R}$. Hence,

$$x' \frac{d\hat{i}}{dt} = x' (\vec{\Omega} \times \hat{i}) \quad (3)$$
and

$$y' \frac{d\hat{j}}{dt} = y' (\vec{\Omega} \times \hat{j}) \quad (4)$$
and

$$z' \frac{d\hat{k}}{dt} = z' (\vec{\Omega} \times \hat{k}) \quad (5)$$

The velocity with respect to the rotating frame is

$$\vec{V}_{x'y'z'} = \frac{dx'}{dt} \hat{i} + \frac{dy'}{dt} \hat{j} + \frac{dz'}{dt} \hat{k} \quad (6)$$

Substituting into (3)(4)(5)(6) into (2) gives

$$\frac{d\vec{r}}{dt} = \frac{dx'}{dt} \hat{i} + \frac{dy'}{dt} \hat{j} + \frac{dz'}{dt} \hat{k} + x' (\vec{\Omega} \times \hat{i}) + y' (\vec{\Omega} \times \hat{j}) + z' (\vec{\Omega} \times \hat{k})$$
or rewriting the cross product terms on the right hand side

$$\frac{d\vec{r}}{dt} = \vec{V}_{x'y'z'} + \vec{\Omega} \times \left(x' \hat{i} + y' \hat{j} + z' \hat{k}\right)$$

and substituting $\vec{r} = x' \hat{i} + y' \hat{j} + z' \hat{k}$ yields

$$\frac{d\vec{r}}{dt} = \vec{V}_{x'y'z'} + \vec{\Omega} \times \vec{r}. \quad (7)$$

Substituting (7) into (2) gives the following equation for the velocity in the inertial frame:

$$\vec{V}_{xyz} = \vec{V}_o + \vec{V}_{x'y'z'} + \vec{\Omega} \times \vec{r}, \quad (8)$$
In words, (8) states:

velocity of $P$ in inertial frame $xyz =$ Velocity of Origin $O' +$ Velocity of $P$ in $x'y'z' +$ velocity term associated with rotating coordinate system.

Differentiating (8) with respect to time yields an equation for the acceleration in the inertial reference frame $xyz$, namely
\[ \ddot{a}_{xyz} = \frac{d\ddot{V}_{xyz}}{dt} = \frac{d\ddot{V}_{Ox} + \ddot{V}_{x'y'z'} + \dddot{\Omega} \times \ddot{r}}{dt} = \frac{d\ddot{V}_{Ox}}{dt} + \frac{d\ddot{V}_{x'y'z'}}{dt} + \frac{d(\dddot{\Omega} \times \ddot{r})}{dt} \]

\[ \ddot{a}_{xyz} = \ddot{a}_{Ox} + \frac{d\ddot{V}_{x'y'z'}}{dt} + \frac{d\dddot{\Omega}}{dt} \times \ddot{r} + \dddot{\Omega} \times \frac{d\ddot{r}}{dt} \]  

(9)

Recall, that \( \ddot{r} \) and \( \dddot{V}_{x'y'z'} \) are measured with respect to the rotating frame. Next, (7) is differentiated (again giving consideration to the unit vectors) to give

\[ \frac{d\dddot{V}_{x'y'z'}}{dt} = \dddot{a}_{x'y'z'} + \dddot{\Omega} \times \ddot{V}_{x'y'z'} \]

(10)

Substituting (10) into (9) yields

\[ \ddot{a}_{xyz} = \ddot{a}_{Ox} + \dddot{a}_{x'y'z'} + \dddot{\Omega} \times \ddot{V}_{x'y'z'} + \frac{d\dddot{\Omega}}{dt} \times \ddot{r} + \dddot{\Omega} \times \frac{d\ddot{r}}{dt} \]  

(11)

Again substituting equation (7) into the last term in (11) gives,

\[ \dddot{\Omega} \times \frac{d\ddot{r}}{dt} = \dddot{\Omega} \times (\dddot{V}_{x'y'z'} + \dddot{\Omega} \times \ddot{r}) = \dddot{\Omega} \times \dddot{V}_{x'y'z'} + \dddot{\Omega} \times (\dddot{\Omega} \times \ddot{r}) \]  

(12)

and substituting (12) into (11) yields,

\[ \ddot{a}_{xyz} = \ddot{a}_{Ox} + \dddot{a}_{x'y'z'} + \dddot{\Omega} \times \dddot{V}_{x'y'z'} + \frac{d\dddot{\Omega}}{dt} \times \ddot{r} + \dddot{\Omega} \times \dddot{V}_{x'y'z'} + \dddot{\Omega} \times (\dddot{\Omega} \times \ddot{r}) \]

\[ \ddot{a}_{xyz} = \ddot{a}_{x'y'z'} + \dddot{a}_{x'y'z'} + \frac{d\dddot{\Omega}}{dt} \times \ddot{r} + 2\dddot{\Omega} \times \dddot{V}_{x'y'z'} + \dddot{\Omega} \times (\dddot{\Omega} \times \ddot{r}) \]  

(13)

<table>
<thead>
<tr>
<th>Term</th>
<th>Physical interpretation of each term</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Absolute rectilinear acceleration of the particle ( P ) relative to the inertial coordinates system ( xyz ).</td>
</tr>
<tr>
<td>II</td>
<td>Absolute rectilinear acceleration of the non-inertial reference frame (( x'y'z' )) relative to the inertial coordinates system ( xyz ). (This will be zero for the earth)</td>
</tr>
<tr>
<td>III</td>
<td>Rectilinear acceleration as measured in the non-inertial reference frame ( (x'y'z') ).</td>
</tr>
<tr>
<td>IV</td>
<td>Tangential acceleration due to angular acceleration of the moving reference frame. (This will also be zero for the earth)</td>
</tr>
<tr>
<td>V</td>
<td>Coriolis acceleration due to motion of a particle in the rotating frame.</td>
</tr>
<tr>
<td>VI</td>
<td>Centripetal acceleration due to rotation of the moving frame.</td>
</tr>
</tbody>
</table>
For the rotating Earth, (13) simplifies to
\[ \vec{a}_{xyz} = \vec{a}_{x'y'z'} + 2\tilde{\Omega} \times \vec{V}_{x'y'z'} + \tilde{\Omega} \times (\tilde{\Omega} \times \vec{r}) \] (14)

The Centrifugal Force and Newtonian Gravity (\( \vec{g}_a \)) are usually combined as:
\[ \vec{g} = \vec{g}_a - \tilde{\Omega} \times (\tilde{\Omega} \times \vec{r}) \] (15)

We will utilize this when we finish deriving the rest of the momentum equation.

The Coriolis Force: \( 2\tilde{\Omega} \times \vec{V}_{x'y'z'} \)
- Results in a curved path in a direction opposite to the direction of coordinate rotation.
- Acts perpendicular to the velocity vector.
- Can only change the direction of travel.

From the derivation given in class, the momentum equation given in an inertial reference frame is
\[ \left( \frac{D\vec{V}}{Dt} \right)_I = -\frac{1}{\rho} \nabla \vec{P} + \nu \nabla^2 \vec{V} + \vec{g}_a \] (16)

Substituting (14) into (16) yields
\[ \left( \frac{D\vec{V}}{Dt} \right)_R = 2\tilde{\Omega} \times \vec{V}_R + \tilde{\Omega} \times (\tilde{\Omega} \times \vec{r}) = -\frac{1}{\rho} \nabla \vec{P} + \nu \nabla^2 \vec{V}_R + \vec{g}_a \]

or
\[ \left( \frac{D\vec{V}}{Dt} \right)_R = -\frac{1}{\rho} \nabla \vec{P} - 2\tilde{\Omega} \times \vec{V}_R + \nu \nabla^2 \vec{V}_R + (\vec{g}_a - \tilde{\Omega} \times (\tilde{\Omega} \times \vec{r})) \] (17)

Substituting (15) into (17) and dropping the \( R \) yields the momentum equation for velocities measured on the rotating Earth.

\[ \frac{D\vec{V}}{Dt} = -\frac{1}{\rho} \nabla \vec{P} - 2\tilde{\Omega} \times \vec{V} + \nu \nabla^2 \vec{V} + \vec{g} \]

Note that these notes are based on the material from R.S. Azad’s “The Atmospheric Boundary Layer for Engineers,” Klewer (1993).