Roots of Polynomials

Ch. 7

Roots of Polynomials

General form:

\[ f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0 \]

- \( n \) = order of the polynomial
- \( a_i \) = constant coefficients

Roots – Real or Complex

1. For an \( n \)th order polynomial – \( n \) real or complex roots
2. If \( n \) is odd \( \Rightarrow \) At least 1 real root
3. If complex roots exist, they are in complex conjugate pairs

\[ \lambda + j\mu \quad \lambda - j\mu \]

\[ i = \sqrt{-1} \]

Roots of Polynomials

Polynomials

- Represent Mathematical models of real systems
- Result from characteristic equations of an ODE
  - The roots of the polynomial are Eigenvalues

Given a Homogeneous ODE:

\[ \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = 0 \]  

(1)

Solution is of the form:

\[ y = e^{rt} \]
Roots of Polynomials

Polynomials
- Represent Mathematical models of real systems
- Result from characteristic equations of an ODE
  - The roots of the polynomial are Eigenvalues

Given a Homogeneous ODE (i.e. dynamic linear system):

\[ \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_2 y = 0 \]  

(1)

Solution is of the form:

\[ y = e^{rt} \]

Substitute into equation (1)

Roots of Polynomials

- The Roots \( r \)'s \( \rightarrow \) Eigenvalues of the system
- Eigenvalues tell us important information about the system behavior.
- Roots represent important Engineering information
- For the Quadratic Case, the eigenvalues tells us:
  - Overdamped – 2 real roots
  - Critically Damped – one real root (the discriminant is zero)
  - Underdamped – 2 complex roots (the discriminant is negative)

Roots of Polynomials

- If only real Roots Exist:
  - Bracketing and Open Methods will work
  - Both methods require initial guesses

- If Roots are Complex
  - Bracketing methods will not work
  - Newton Raphson will work if the language handles complex numbers. (Still susceptible to diverging)
  - Introduce several New methods that avoid these problems.
Mueller’s Method

• Project a parabola through 3 points on the function

• Need to find the coefficients that force the parabola through the 3 points

• Use the coefficients & quadratic formula to find where the parabola intersects the x-axis → Root Estimate

\[ y = a_0 + a_1 x + a_2 x^2 \]

Mueller’s Method

• Given an Equation for a parabola

\[ y = a_0 + a_1 x + a_2 x^2 \]

• Rewrite relative to the point \( x_2 \)

\[ f(x) = c + b(x - x_2) + d(x - x_2)^2 \]

• Substitute the 3 points \((x_0, x_1, x_2)\) into the General equation

\[
\begin{align*}
(1) & \quad f(x_0) = c + b(x_0 - x_2) + d(x_0 - x_2)^2 \\
(2) & \quad f(x_1) = c + b(x_1 - x_2) + d(x_1 - x_2)^2 \\
(3) & \quad f(x_2) = c + b(x_2 - x_2) + d(x_2 - x_2)^2 = c
\end{align*}
\]

3 Equations & Unknowns

Mueller’s Method

• Substitute the value of \( c \) into (1) & (2)

\[
\begin{align*}
(4) & \quad f(x_0) - f(x_1) = b(x_0 - x_1) + d(x_0 - x_1)^2 \\
(5) & \quad f(x_1) - f(x_2) = b(x_1 - x_2) + d(x_1 - x_2)^2
\end{align*}
\]

• Solve the 2 Equations and 2 unknowns

• First, Make the following changes of variables to help us solve the equations

\[
\begin{align*}
h_0 &= x_0 - x_2 \\
h_1 &= x_1 - x_2 \\
h_2 &= x_2 - x_0 \\
p_0 &= f(x_0) \\
p_1 &= f(x_1) \\
p_2 &= f(x_2)
\end{align*}
\]

\[
\begin{align*}
h_0 &= x_0 - x_2 \\
h_1 &= x_1 - x_2 \\
h_2 &= x_2 - x_0 \\
p_0 &= f(x_0) \\
p_1 &= f(x_1) \\
p_2 &= f(x_2)
\end{align*}
\]
Mueller’s Method

- Next, add and subtract \( f(x_j) \) to the LHS (4) and add and subtract \( x_j \) to the RHS of (4)

\[
[f(x_j) - f(x_j) + f(x_j) - f(x_j)] + \delta[x_j - x_j] + a[x_j - x_j] + \beta[x_j - x_j]^2
\]

\[
f(x_j) - f(x_j) = h(x_j - x_j) + a(x_j - x_j)^2
\]

\[
 h_0 = x_j - x_0 \quad \delta_j = f(x_j) - f(x_0) \over x_j - x_0
\]

\[
 h_1 = x_j - x_1 \quad \delta_1 = f(x_j) - f(x_1) \over x_j - x_1
\]

Simplifying:

Solve for \( a \) and \( b \):

\[
a = \frac{\delta_j - \delta_1}{h_1 + h_0}
\]

\[
b = ah_1 + \delta_1
\]
Now that we know \( a, b \) and \( c \) return to our equation for a parabola:

\[
f(x) = c + bx - x^2 + ax^3 = 0
\]

And solve for our root estimate \( x_3 \) using the Quadratic Formula

\[
f(x) = c + bx - x^2 + ax^3 = 0
\]

Mueller’s Method

Use alternative form of the Quadratic formula to reduce round-off error

\[
x_0 = x_3 + \frac{-2c}{b + \sqrt{b^2 - 4ac}}
\]

Approximate Root (Current Estimate) Previous Root Estimate

Mueller’s Method—Error & Implementation Strategies

Relative Percent Error:

\[
\varepsilon = \left| \frac{x_n - x_{n-1}}{x_n} \right| \times 100
\]

- Choose the sign in the denominator to agree with the sign of \( b \) \( \rightarrow \) result largest denominator \( \rightarrow \) \( x_i \) will be the root estimate closest to \( x_2 \)
- 2 Strategies for discarding 1 of the \( x \)’s when moving on to the next iteration
  1. Take the original 2 points closest to \( x_i \)
  2. Replace \( x_x, x_y, x_z \) with \( x_1, x_2, x_3 \)
    - Best when complex roots are needed
Mueller’s Method Example

- Find the roots of
  \[ f(x) = x^3 - x^2 - 5x - 3 \]

Mueller’s Method - Notes

- Compared to Newton-Raphson, Mueller’s method only requires function values, NOT derivatives.
- Will find complex roots
- Mueller’s method can be used to find complex roots.
- Mueller’s method fails when \( f(x_1) = f(x_2) = f(x_3) \)
- Rate of convergence is slightly less than quadratic.
- Can diverge

Review of Root Finding Methods

- **Bracketing Methods**
  - Graphical
  - Bisection
  - False Position
  - 2 Guesses needed
  - 2Always Converge
  - Slow Convergence

- **Open Methods**
  - Successive Iteration
  - Newton Raphson
  - Modified Newton Raphson
  - Secant
  - 1 Guess needed
  - Possible Divergence
  - Rapid Convergence
Review of Root Finding Methods

- Roots of Polynomials
  - Muller’s Method (can handle imaginary roots)
- 3 Guesses needed
- Possible Divergence
- Rapid Convergence